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APPLICATION OF THE DIRECT METHOD OF LIAPUNOV TO THE POWER SYSTEM STABILITY PROBLEM

by

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I. INTRODUCTION

Since the early days of alternating current electric power generation and utilization, oscillations of power flow between synchronous machines have been known to be present. Theoretically, at least in the steady state, no such oscillations would exist. However, no real power system is truly in the steady state as there are always disturbances of small or large size which cause the system to be continually adjusting to meet new conditions.

As time progressed the power systems grew in size and complexity with large distances between the various generating plants and load areas. With the transmission of larger blocks of power over longer distances it was found that portions of systems would tend to lose synchronism with each other when large disturbances due to faults or other causes occurred. It might be said that such problems could be avoided by keeping transmission distances short and isolating various areas of generation, but economic studies have shown the large interconnected system to be desirable.

The study of the stability of power systems under transient conditions is a tedious task because the differential equations describing even the simplest system are non-linear. Studies of large systems include numerous involved calculations. The general approach has been to obtain a time solution and to observe if the various machines tend to lose or maintain synchronism.

The equal-area criterion for stability is a graphical method for presenting the stability problem for a two-machine system, but it does not give time response directly, nor does it apply to a system of three or more machines which may swing independently.

The phase plane method applies to the two-machine problem, but it cannot be extended to three or more machines. Neither does it give the time response directly. It is useful in defining stability limits for two machines.

In this dissertation it will be shown how Liapunov functions can be applied in a direct method of solution. This method emphasizes the idea of the relative stability of the system, but does not develop the time response. The close relationship of the equal-area criterion, the phase plane method, and the method of Liapunov functions will be demonstrated for two machines. The method of Liapunov functions has the advantage of being applicable to three or more machines. It will be shown that the method is exact for two machines, but somewhat conservative for three or more machines.

Regardless of the previous history of occurrences of faults or switching operations, the method using Liapunov functions is applicable from the time of the last operation. The angular positions and the angular velocities of the machines must be determined as a function of time up to this last operation, and then the method described may be applied to determine the stability of the system with these initial conditions. The method will then relieve the necessity of further point-by-point calculations of swing curves, and if the method predicts stability there should be no doubt that stability can be achieved since the method is presumed to be conservative.

II. REVIEW OF LITERATURE

Early in the history of the utilization of alternating current electric power it was realized that there was an upper limit to the amount of power which could be transmitted over a given circuit. If the power is increased in small increments with sufficient time between increments to allow the small oscillations to die out, the power reaches a limit called the steady state stability limit. Of course the later increments would have to approach an infinitesimally small value since a finite step could put the system in the unstable region. In fact, if the power is always increased by finite amounts, that power which can be transmitted without instability resulting when the power increment is applied represents the transient stability limit. The transient stability limit is thus normally below that of the steady state. Crary (1) (2) and Kimbark (3) (4) give excellent discussions of the various aspects of stability.

When the stability limit is exceeded, synchronism is lost between the various parts of a system. In a simple system consisting of a generator supplying a motor through a transmission line, the motor will slow down or stop when the two machines fall out of step or lose synchronism. In a larger, more complex system the individual parts of the system may continue to operate independently if there is sufficient generation in each area to carry the local load. Usually, however, large energy sources are cut off from the system they are supplying and the whole network tends to fall apart.

The increasing dependence of industry and the general public on continuity of electric service has led to efforts to maintain service under almost every condition of operation. Stability studies have enabled the prediction of system behavior under transient conditions due to faults or normal switching operations and thus to insure continuity of service under all except the most unusual conditions of system operation.

The first careful study of power system stability was made by Bergvall and Evans in 1924 (5). Behavior of an artificial line under both steady load and transient conditions was studied. Effects of synchronous condensers at various points along the line were observed. The transients were recorded using an oscillograph.

In 1926 Griscom (6) proposed using a mechanical analogy to study the transient stability problem in electric power transmission. Shortly thereafter Bergvall and Robinson (7) built a mechanical model of the Conowingo transmission system using springs and rotating arms. This model of a seven machine system gave results which agreed favorably with the actual conditions and also created a great deal of interest because it gave a graphic presentation of machine behavior during transients.

The realization that the rapid system growth taking place would increase the magnitude of the stability problem encouraged a great many contributions to be made during the late twenties and early thirties. The factors affecting stability and a summary of methods of analysis then in use were published in a report by the A.I.E.E. Subcommittee on Interconnection and Stability Factors (8). The report includes an extensive bibliography of pertinent articles.

Among the more important contributions of the period were the papers by Fortescue (9), Wilkins (10), Evans and Wagner (11), Park and Bancker (12), Longley (13), Summers and McClure (14), and Byrd and Pritchard (15). Fortescue, 1925, and Park and Bancker, 1929, made analytical studies using the equal-area criterion, the step-by-step method and time angle curves. Park and Bancker also published the first set of universal swing curves. Wilkin's paper published in 1926 described the first transient stability tests on an actual system made on the 220 KV lines of the Pacific Gas and Electric Company. The same year Evans and Wagner made analytical step-by-step calculations of the work described in Wilkin's paper with good agreement between actual and calculated results.

Longley's 1930 paper was a very detailed discussion of the step-by-step method of calculating swing curves for the purpose of acquainting other engineers with the method. He also went into the effects of excitation on stability. In the same year Summers and McClure extended the work of Park and Bancker to include the effect of resistance in their precalculated swing curves. They also made a summary of the effect of different types of apparatus on stability and concluded that high speed breakers and relays were essential. In 1933 Byrd and Pritchard published an extensive set of curves which gave the maximum time to clear the fault without losing synchronism in a two-machine system. The curves were given for a range of initial loads and network conditions.

Later contributions to the two-machine stability problems were made by Skilling and Yamakawa (16) in 1940 and Fouad (17) in 1956. Skilling and Yamakawa developed a method of calculating swing curves using graphical integration of the power angle curves. Fouad's dissertation contains an extensive set of universal swing curves which include the effect of resistance and allow for multiple switching of circuits.

The increasing complexity of electric power systems led to the development of calculating machines to speed the work and reduce the labor of computation. One of the first was a d-c board for short circuit calculations reported in the General Electric Review (18) in 1916 and Lewis (19) published a description of an improved model in 1920. Gray (20) describes the first artificial system built in 1917 to simulate a complete power system. These early ideas led to the development of the a-c network analyzer at Massachusetts Institute of Technology as presented by Hazen, Schurig and Gardner in 1930 (21). Parker (22) pointed out in his paper the improvements made in the a-c network calculator up to 1941. Methods of speeding up the calculations on the network analyzer were developed by Heffron and Rothe (23), 1955, who describe the operation of an automatic calculator for the incremental angles and Wood (24) who first used a simple computer for the same purpose in 1956. The a-c network analyzer has been the basic computer used in system studies for many years, but recently the large digital machine has been used more and more for this work.

The differential analyzer was used extensively for making studies particularly involving voltage regulation and excitation systems. Kuehni and Peterson (25), 1944, describe an electro-mechanical machine built at General Electric. In 1950 Cook, Kirchmayer and Weygandt (26) give considerable detailed information in using an automatic curve follower to generate functions and perform multiplications of scalar and vector quantities. Concordia (27) presented a paper in Paris in 1950 describing the use of the differential analyzer in system studies with particular emphasis on the problem of power transmission over a long line taking into account the action of the generator excitation system.

Analogue methods found quite wide use in studying the problem of stability. Already mentioned was the work of

Bergvall and Robinson (7) with the mechanical simulation of a power system. In 1951 Boast and Rector (28) used an RLC circuit in conjunction with a photoformer in an ingenious arrangement to generate the non-linear differential equation of a machine and its associated line. The circuits could be coupled together to represent multimachine systems. A year later Kaneff (29) in England proposed a system using frequency modulated oscillators to represent machines with the modulation depending on the difference between the output and the input of a machine.

A paper published in 1957 by Shackshaft and Aldred (30) described the use of a special electronic analogue computer for stability calculations. The equations solved by this computer were based on Park's analysis (31) and included the effects of saliency, field time constants and excitation systems. The output gave swing curves which showed the effect of clearing time on synchronous machine transient stability. Aldred and Doyle (32) used an electronic analogue computer to make stability studies in 1957 which included the effects of damping. They were able to demonstrate instability on subsequent swings under certain system conditions.

The Russians made extensive use of analogue computers in making stability studies. A paper published in 1961 by Sokolov, Gurevich and Khvoshchinskaya (33) describes a computer setup which could handle a large complex system including the effect of voltage regulation. A year later Gruzdev, Kuchumov, Luginsky, Sokolov and Venikov (34) reported on the use of various types of computers in studying transient stability with emphasis on the uses of analogue computers. It was also mentioned that Liapunov's direct method might be an aid in stability studies.

Although not much used in this country, the use of miniature models of systems has found considerable use in foreign countries. Robert (35), 1950, described a method of representing whole systems in miniature by using machines and other equipment having scaled down voltages, inertias, etc. The "microréseaux" were developed and used by Électricité de France. A typical stability run would require only a few minutes. Venikov (36) presented a paper in 1952 which described the wide use of the miniature model in Russia. The work in the two countries is similar but the Russian equipment seems to be more crude and bulky in appearance.

As digital computers became available to the engineer, investigations began to be made as to their use in solving the transient stability problem. An early study was made by Johnson and Ward (37) in 1956. They compared the step-by-step method used with the network analyzer and the Runge-Kutta method of solving the equations for the one and three machine cases showing that the digital computer method could be more accurate even with rather large time increments.

Gabbard and Rowe (38), 1957, describes an early digital program which replaced a network analyzer program. The work on the analyzer was taking so much time that it could not be completed soon enough to be of real use. In 1958 Lane, Long and Powers (39) used a special partitioning of the admittance matrix to shorten a computer program based on Gill's variation of the Runge-Kutta method. Further improvements were made to shorten computer time by Rindt, Long and Byerly (40) during the following year.

In 1959 Stagg, Gabrielle, Moore and Hohenstein (41) used the Gill variation in a program on a large machine and were able to represent 50 induction and/or synchronous machines, 200 busses and 300 lines and/or transformers with automatic simulation of a desired fault and clearing sequence. Similar work was done in the same year by Dyrkacz and Lewis (42) who retained the topological structure of the system so that bus voltages and line flows could be obtained readily. One drawback was that only symmetrical faults or switching could be accommodated. This program was greatly expanded the following year by Dyrkacz, Young and Maginniss (43) so that it was possible to accommodate up to 96 machines and to include the effects of saliency, regulation, excitation and speed governor action. Loads could be represented in several different ways. A somewhat special program was devised by Johannesen and Harle (44) in 1961 to determine limiting operating curves for a single machine operating into an infinite bus under various conditions of loading, excitation and fault location.

A somewhat different approach to the two-machine stability problem was taken by Goodrich (45) and Rao (46). Goodrich in 1952 actually developed and used phase plane diagrams although he did not call them by that name nor did he exploit their properties fully. He also developed generalized swing curves which could be used under conditions of reclosing. His work did not seem to receive much attention for Rao, 1962, developed the same phase plane diagrams and was hailed for taking a new approach to the stability problem. He too failed to exploit the phase plane approach fully as will be indicated in a later portion of this dissertation.

III. APPLICATION OF THE DIRECT METHOD OF LIAPUNOV TO THE ONE OR TWO-MACHINE SYSTEM

When the power system to be studied consists of one or a group of several generators connected to a large system through a long transmission line or lines, the system may be considered as a single equivalent machine connected to an infinite bus through a transmission line. If the system consists of two machines or groups of machines tied through a long transmission line, the network may be replaced by an equivalent generator connected through the line to an infinite bus as shown by Kimbark (3, p. 132). It is evident that the one and two-machine systems can be treated as identical problems in the mathematical procedure which follows.

For the remainder of this dissertation the following assumptions are made for the electrical system under study.

1. A synchronous machine may be represented by a constant voltage back of its transient reactance.
2. Flux linkages in the rotor circuit of a synchronous machine are constant.
3. Torque input to the synchronous machine is constant.
4. Angular momentum of the synchronous machine is constant.
5. The resistance of lines and machines is neglected.
6. All damping torques are neglected.

The chief reason for making the above assumptions is that the calculations are very much simplified and thus attention can be focused on the main problem of determining stability. The above group has been used in whole or in part in many transient stability studies in the past.

The swing equation for the single machine system is, according to Kimbark (3, p. 124),

$$M \frac{d^2\delta}{dt^2} = P_a = P_i - P_u \qquad 1$$

where δ is the angle between the machine voltage and the infinite bus, M is the inertia constant, P_a the accelerating power, P_i the power input and P_u the power output. P_u is given by the power-angle relation as given by Kimbark (3, p. 135)

$$P_u = E_1 E_2 Y_{12} \sin \delta \qquad 2$$

where E_1 is the generator voltage back of its reactance, E_2 is the infinite bus voltage and Y_{12} is the combined susceptance of the generator and line. Equation 2 may be written as

$$P_u = P_m \sin \delta \qquad 3$$

Substitution of equation 3 in 1 leads to the swing equation in a different form

$$M \frac{d^2\delta}{dt^2} = P_i - P_m \sin \delta \qquad 4$$

The equal-area criterion essentially as presented by Kimbark (3, p. 122ff) is now examined for the one-machine case and the results obtained from it are to be shown as being identical with results using the phase plane approach or the direct method of Liapunov. The system under consideration is shown in Fig. 1a. Its power-angle curves are shown in Fig. 1b for conditions before, during and after the fault. The horizontal line denoted $P = P_i$ represents the power input to the generator. Before occurrence of the fault the machine was operating at the intersection of the input line and the pre-fault power-angle curve. Analysis of these curves by the equal-area criterion reveals that the machine will begin swinging from $\delta = \delta_1$ to $\delta = \delta_m$, but will remain in synchronism with the infinite bus because area A_2 can be made as large as area A_1. The limiting case of stability would be illustrated by Fig. 1c in which A_2 has a maximum value when $\delta = \delta_m$ at the intersection of the post-fault curve with the power input line. If the generator were to swing to this maximum value, $\delta = \delta_m$, and remain stable, it would then swing back and approach $\delta = \delta_o$ as a minimum angle. Of course, a more severe disturbance than that illustrated in Fig. 1b would be necessary to cause the machine to swing to $\delta = \delta_m$. In general, δ_o would not be the same as δ_1 where the oscillation began. Fig. 1c, with $A_1 = A_2$, serves to define δ_o as the minimum angle to which a machine would swing if it had gone to the maximum angle, $\delta = \delta_m$, on the previous half cycle of oscillation.

If both sides of equation 1 are multiplied by $\frac{2}{M} \frac{d\delta}{dt}$ and an integration performed, the result is

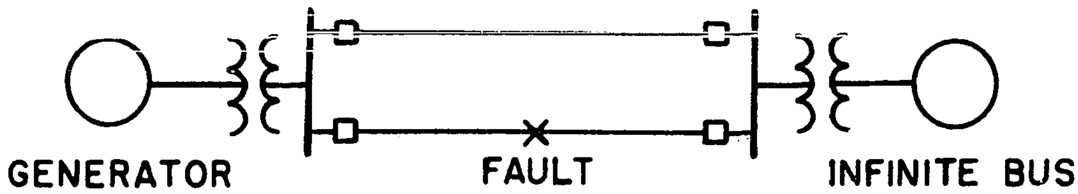

Fig. 1a. One-machine system

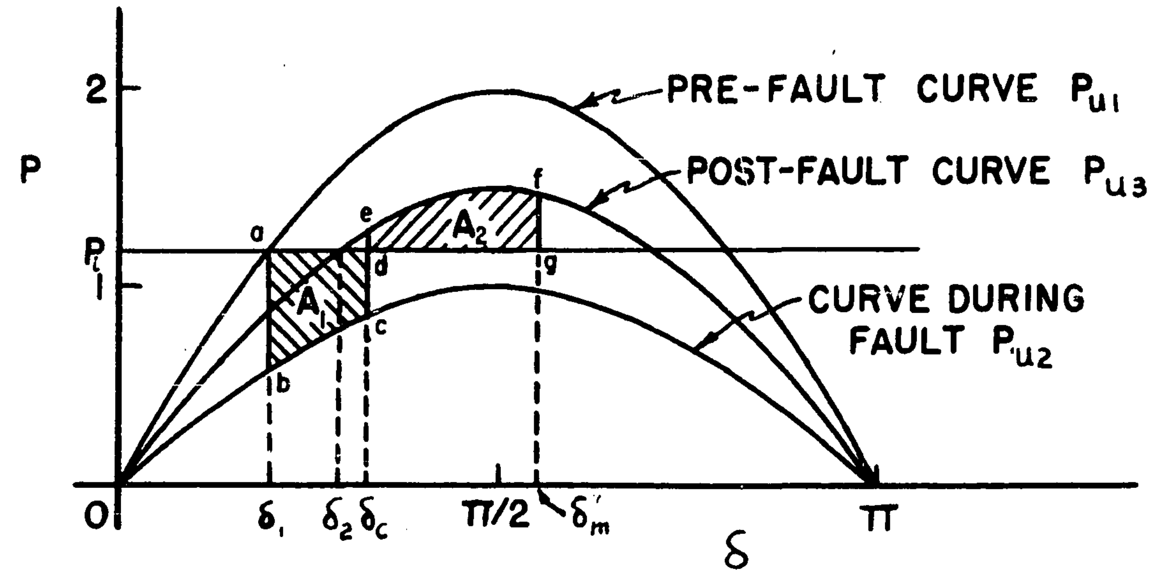

Fig. 1b. Power-angle curves for a fault with subsequent clearing

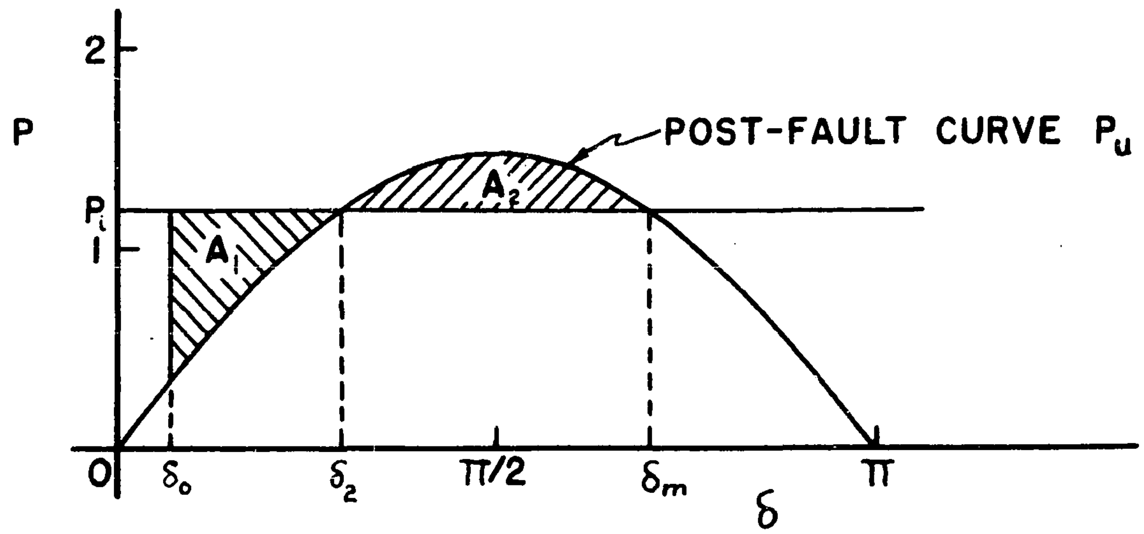

Fig. 1c. Power-angle curve for the post-fault condition

$$\left(\frac{d\delta}{dt}\right)^2 = \frac{2}{M}\int_{\delta_o}^{\delta} P_a d\delta' \qquad 5$$

When the generator is swinging with respect to the infinite bus from δ_o to δ_m in Fig. 1c, the condition for stability is that $d\delta/dt$ must be zero at $\delta = \delta_m$ and thus the right hand side of equation 5 must be zero or

$$\int_{\delta_o}^{\delta_m} P_a d\delta = \int_{\delta_o}^{\delta_m} (P_i - P_u) d\delta = 0 \qquad 6$$

It is thus evident that area A_1 must equal A_2 in Fig. 1c if a stable condition is to obtain. Stability will be maintained if δ stays between δ_o and δ_m, and in a case with damping δ is expected to approach δ_2.

Minorsky (47, p. 118ff) studied the problem of the simple pendulum with a constant torque applied. The mathematical equations for his pendulum problem are the same as for the electrical system under study here. He mentioned the interesting conclusion that the time to reach δ_m in the critical case is infinitely long.

If in equation 5 P_a is set equal to $P_i - P_m \sin \delta$ and the indicated integration is carried out, the result is

$$\left(\frac{d\delta}{dt}\right)^2 = \frac{2}{M}\left[P_i(\delta - \delta_o) + P_m(\cos \delta - \cos \delta_o)\right] \qquad 7$$

Following the work of Thaler and Pastel (48, p. 65ff), a similar result may be obtained in the phase plane. To make a phase plane plot, it is necessary only to plot velocity versus displacement. The plot is called a phase trajectory. The trajectories for the velocity and displacement of a mass constrained by a linear spring with no damping would be a set of ellipses centered on the origin if displacement is measured from the equilibrium position. Thus, for the simple harmonic motion of the system, velocity has a maximum positive or negative value when the displacement is zero and the displacement has a maximum positive or negative value when the velocity is zero.

The ellipses would become larger as the total energy of the system increased. Since the total energy may take on any value, the family of trajectories would cover the whole phase plane. The term phase plane is used because each point on a given trajectory describes a particular state or phase of the system.

To obtain the equation of a trajectory for the one-machine system being considered here, equation 4 is rewritten by letting

$$\frac{d\delta}{dt} = \omega \qquad 8$$

$$\frac{d^2\delta}{dt^2} = \frac{d\omega}{dt} \qquad 9$$

Thus equation 4 becomes

$$M\frac{d\omega}{dt} = P_i - P_m \sin\delta \qquad 10$$

If equation 10 is divided by the corresponding sides of equation 8 the result is

$$M \frac{d\omega}{d\delta} = \frac{P_i - P_m \sin \delta}{\omega} \qquad 11$$

Equation 11 may be integrated to give

$$\frac{M}{2} \omega^2 = P_i(\delta - \delta_o) + P_m(\cos \delta - \cos \delta_o) \qquad 12$$

If the substitution $\omega = d\delta/dt$ is made in equation 12 and both sides multiplied by 2/M, the result is identical with 7. If equation 12 is plotted in the phase plane with δ_o at its minimum value as shown in Fig. 1c, the result is a trajectory in the ω-δ plane as shown in Fig. 2a and labeled as the maximum trajectory. This trajectory is distorted from an ellipse because the "spring" is non-linear. The trajectory does not surround the origin, but a simple change of variable which is made later will move the origin to $\delta = \delta_2$. δ_2 is the stable equilibrium point for the system as explained in connection with Fig. 1c. If the ω axis is shifted to $\delta = \delta_2$ in Fig. 2a and the trajectories are examined, it is evident that the velocity, $\omega = d\delta/dt$, has a maximum positive or negative value when the displacement is zero at $\delta = \delta_2$ and the displacement from the equilibrium is a maximum positive or negative value when the velocity is zero. The behavior of this system when stable is thus similar to the simple spring mass system. If the system δ exceeds δ_m the "spring" reverses its force and causes the velocity to increase rather than decrease and the machine loses synchronism.

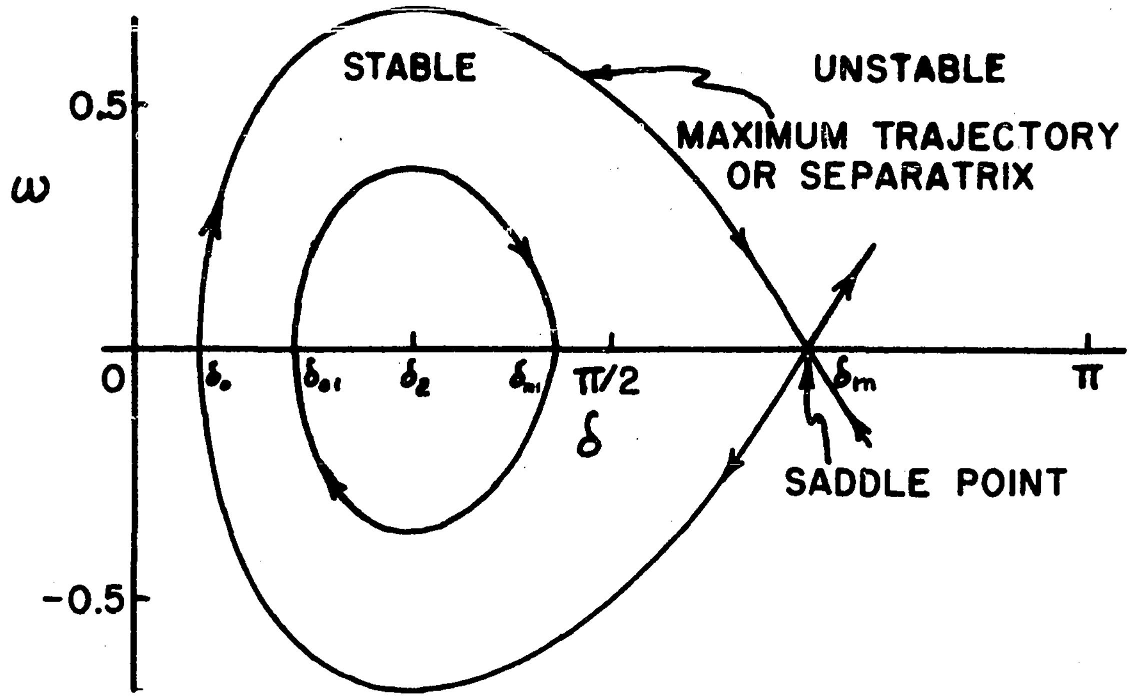

Fig. 2a. Phase trajectories in the ω-δ plane, post-fault conditions

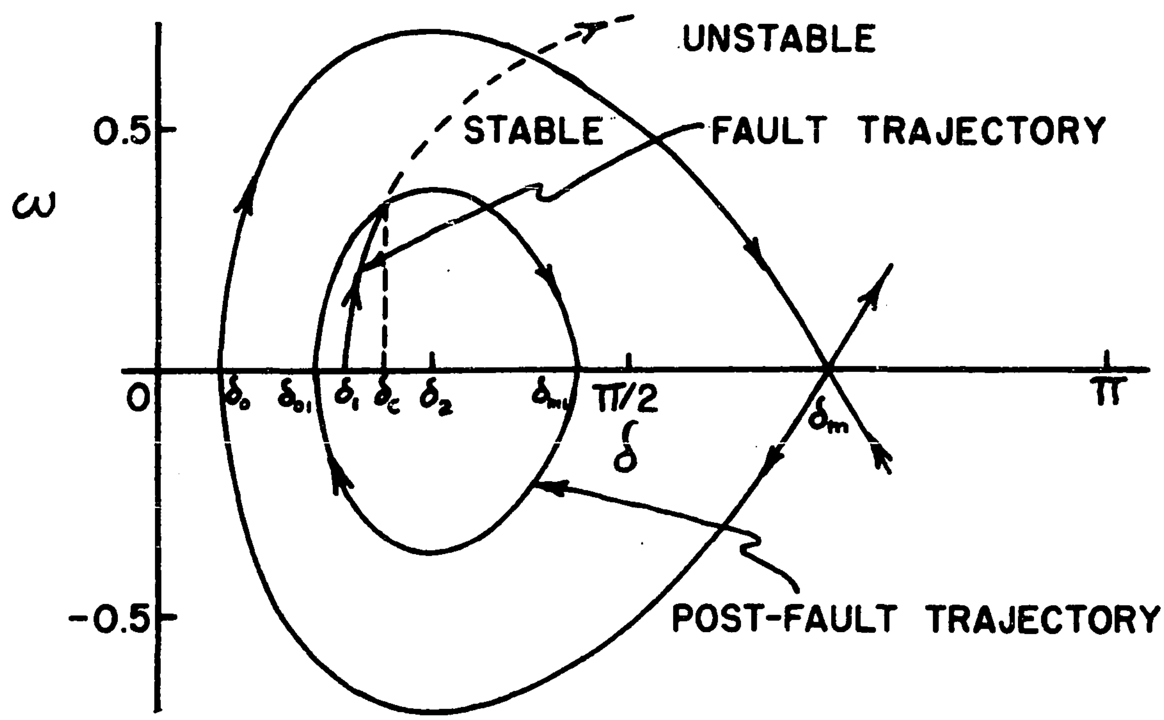

Fig. 2b. Phase trajectories under fault and post-fault conditions

The direct method of Liapunov may be applied to the one-machine problem by following a procedure suggested by Graham and McRuer (49, pp. 358-361). Before the Liapunov function is written the origin is shifted to the stable equilibrium point at $\delta = \delta_2$ in Fig. 1c. In general, δ_2, the stable equilibrium value of δ, is a constant for any given set of system parameters. Equilibrium or singular points are at values of the variables such that equation 11 becomes the indeterminant form

$$\frac{d\omega}{d\delta} = \frac{0}{0} \qquad 13$$

Thus one singular point exists at $\omega = 0$, $\delta = \delta_2$ and a second at $\omega = 0$, $\delta = \delta_m$ since $P_i = P_m \sin \delta$ for the given values of δ. When $\delta = \delta_2$ the equilibrium is stable and when $\delta = \delta_m$ it is unstable.

The shift to the stable equilibrium point is made by the convenient change of variable $x = \delta - \delta_2$. Equation 10 then becomes

$$M \frac{d\omega}{dt} = P_i - P_m \sin(x + \delta_2) \qquad 14$$

The right hand sides of equations 10 and 14 are identical since $\delta = x + \delta_2$ and $d\delta/dt = d(x + \delta_2)/dt = dx/dt = \omega$. As δ_2 is a constant, defined above, its derivative with respect to time is zero.

The positive definite Liapunov function* is chosen as

*The Liapunov function and quadratic forms are defined in the appendix.

$$V = \frac{\omega^2}{2} + \frac{1}{M}\int_0^x \left[-P_i + P_m \sin(x' + \delta_2)\right] dx' \qquad 15$$

with

$$x\left[-P_i + P_m \sin(x + \delta_2)\right] = 0, \; x = 0 \qquad 16$$

$$x\left[-P_i + P_m \sin(x + \delta_2)\right] > 0, \; 0 < x < (\delta_m - \delta_2)$$
$$0 > x > -(\delta_2 + \pi) \qquad 17$$

The range of validity of the Liapunov function thus corresponds to the region over which the restoring force is positive. Equation 15 is a positive definite function over the interval specified since such a function is defined as equal to zero at the origin and greater than zero at all other points in the region about the origin.

As explained in the appendix, the Liapunov function is generally chosen as a quadratic form in the variables of the system or as a quadratic form plus an integral in many cases. Unfortunately the exact function to choose is not known in advance. The procedure is to try various functions until one is found which meets the conditions imposed on dV/dt enumerated in the appendix. Experience is helpful in this process.

A great deal of experimentation with various Liapunov functions took place before the proper form of equation 15 was discovered. With the form of equation 15 established, it was noted that the integrand was equal to the negative of the acceleration in equation 4 if a change of variable was made. This experience proved helpful in determining Liapunov functions for the three-machine systems studied later.

The system is stable for the equilibrium point under consideration if the total derivative of V with respect to time is at least negative semi-definite. A function is negative semi-definite if it is equal to zero at the origin and less than or equal to zero in a region about the origin. The time derivative of equation 15 is

$$\frac{dV}{dt} = \dot{V} = \omega \frac{d\omega}{dt} + \frac{1}{M}\left[-P_i + P_m \sin(x + \delta_2)\right] \frac{dx}{dt} \qquad 18$$

or, since $d\delta/dt = dx/dt$ and making substitutions from equations 8 and 14, 18 becomes

$$\dot{V} = \frac{dx}{dt} \frac{1}{M}\left[P_i - P_m \sin(x + \delta_2)\right]$$

$$+ \frac{1}{M}\left[-P_i + P_m \sin(x + \delta_2)\right] \frac{dx}{dt} \qquad 19$$

or

$$\dot{V} \equiv 0$$

Thus $\dot{V}$ is negative semi-definite and the equilibrium is stable.

The Liapunov function may be compared with equation 7 in the following manner. If the integration indicated in 15 is performed, the result is

$$V = \frac{\omega^2}{2} + \frac{1}{M}\left[-P_i x - P_m \cos(x + \delta_2) + P_m \cos \delta_2\right] \qquad 20$$

Substituting $\omega = d\delta/dt$, $x = \delta - \delta_2$ and rearranging, equation 20 becomes

$$\left(\frac{d\delta}{dt}\right)^2 = \frac{2}{M}\left[P_i(\delta - \delta_2) + P_m(\cos \delta - \cos \delta_2)\right] + 2V \qquad 21$$

or, after adding and subtracting $P_i\delta_o + P_m \cos \delta_o$ and re-arranging,

$$\left(\frac{d\delta}{dt}\right)^2 = \frac{2}{M}\left[P_i(\delta - \delta_o) + P_m(\cos \delta - \cos \delta_o)\right] + 2V$$

$$+ \frac{2}{M}\left[P_i(\delta_o - \delta_2) + P_m(\cos \delta_o - \cos \delta_2)\right] \qquad 22$$

Equations 7 and 22 are identical if

$$V = \frac{1}{M}\left[P_i(\delta_2 - \delta_o) + P_m(\cos \delta_2 - \cos \delta_o)\right] \qquad 23$$

The three methods of approach then give the same equation for the phase trajectory of the system.

The maximum trajectory, which is the stability boundary for the system, may be found by setting $\omega = 0$ and $x = \delta_m - \delta_2$ in equation 20. The result is a value for V which defines precisely the region of stability for the equilibrium at $\delta = \delta_2$. The same trajectory was found using equation 12 which was developed by the phase plane approach. Fig. 2a illustrates stable and unstable regions in the ω-δ plane and also shows the behavior at the unstable equilibrium, $\delta = \delta_m$, which is a saddle point. The plot for the ω-x plane is identical in shape with the vertical axis at $\delta = \delta_2$.

If the system of Fig. 1a is initially operating at $\delta = \delta_1$, and a fault occurs, the trajectory will take the form shown in Fig. 2b. If the fault is cleared in time, operation will be stable along a curve as indicated by the solid closed curve within the stable boundary. For a sustained fault or a longer delay in clearing time the faulted trajectory would enter the unstable region as shown by the dotted curve in Fig. 2b. Cases in which multiple switching occur are handled in much the same way as for the single instant of switching except that a series of connected trajectories will be obtained. Calculation of time along the trajectory is described below.

It is not necessary to plot the curves shown in order to determine stability. The range of values for V are from 0 at $x = 0$, $\omega = 0$ to a maximum at $\omega = 0$, $x = \delta_m - \delta_2$. If substitution in equation 20 of the values of ω and x at the last instant of switching results in a value less than the maximum stable value, the system is stable with the relative degree of stability indicated by whether V is nearer to its minimum or maximum value. An unstable condition is indicated if $x \geq (\delta_m - \delta_2)$ or V is greater than the maximum stable value for $0 < x < (\delta_m - \delta_2)$.

Time may be computed along a trajectory by a method suggested by Thaler and Pastel (48, pp. 82-83). The time needed to traverse a small segment of a trajectory is obtained by taking the ratio of the average velocity to the change in displacement represented by the segment. Total time is determined by summing the incremental times for a suitable number of segments between the points of interest along a trajectory. The critical clearing time may be computed in this manner by using the trajectory for the faulted condition.*

*An example is illustrated in Fig. 4b.

Although Goodrich (45) and Rao (46) discussed the stability problem in the phase plane, neither of them introduced the approach using the direct method of Liapunov. Rao developed the equation for the total energy in the one-machine system, but he apparently did not realize that it was a Liapunov function. The advantage of the Liapunov method is that it may be readily extended to the case of three machines or more whereas the phase plane approach is not very useful when there are more than two machines involved.

IV. EXTENSION OF THE METHOD TO THE THREE-MACHINE CASE

The direct method of Liapunov may be applied to any set of differential equations which satisfy the conditions set down in the appendix. Since the equations for a three-machine system as described satisfy the conditions, the direct method will now be applied to such a system. The result of the application of the method will be the development of a surface in the phase space which will serve as a boundary between the regions of stability and instability of the system just as the maximum trajectory did in the phase plane.

The first three-machine system to be considered will be one which can be studied in the phase plane because of its symmetry. Although a rather special case, it is interesting to compare its behavior with that of a two-machine system. In addition, the special case serves as a transitional step to the more general system of three machines.

The equations for the system illustrated in Fig. 3a take the form

$$M_1 \frac{d^2\delta_1}{dt^2} = P_1 - E_1E_2Y_1\sin(\delta_1 - \delta_2) - E_1E_3Y_3\sin(\delta_1 - \delta_3)$$

$$M_2 \frac{d^2\delta_2}{dt^2} = P_2 - E_1E_2Y_1\sin(\delta_2 - \delta_1) - E_2E_3Y_2\sin(\delta_2 - \delta_3)$$

$$M_3 \frac{d^2\delta_3}{dt^2} = P_3 - E_1E_3Y_3\sin(\delta_3 - \delta_1) - E_2E_3Y_2\sin(\delta_3 - \delta_2) \qquad 24$$

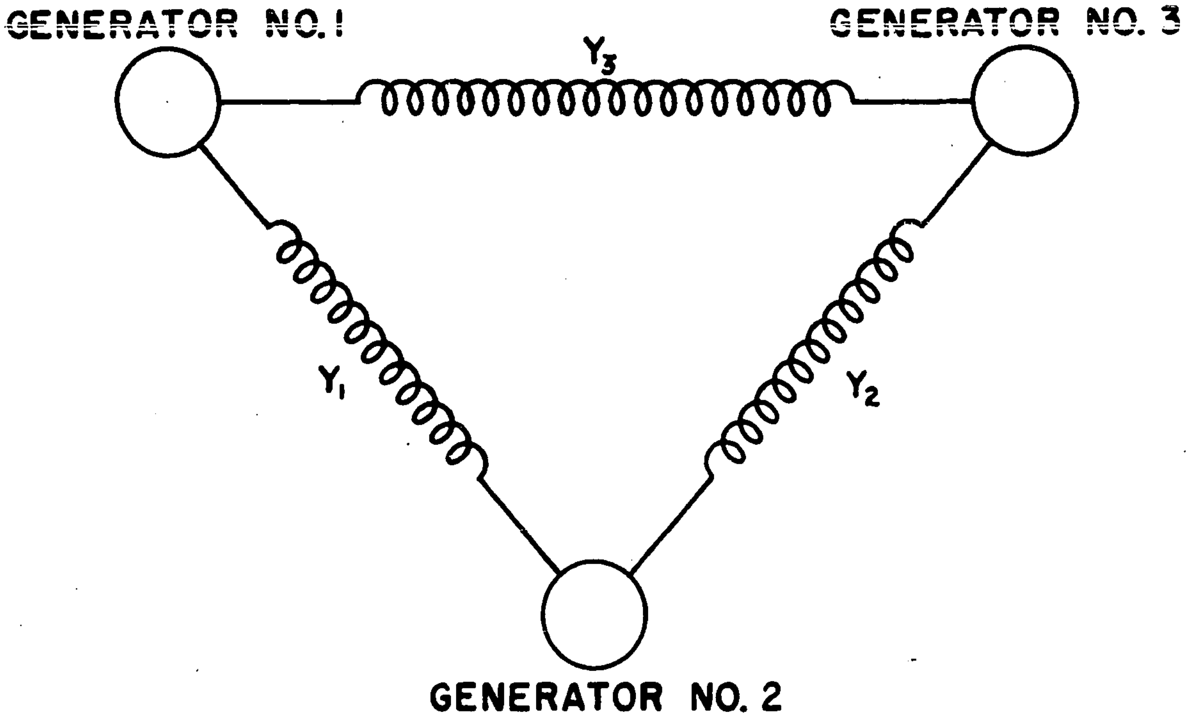

Fig. 3a. Simplified three-machine system

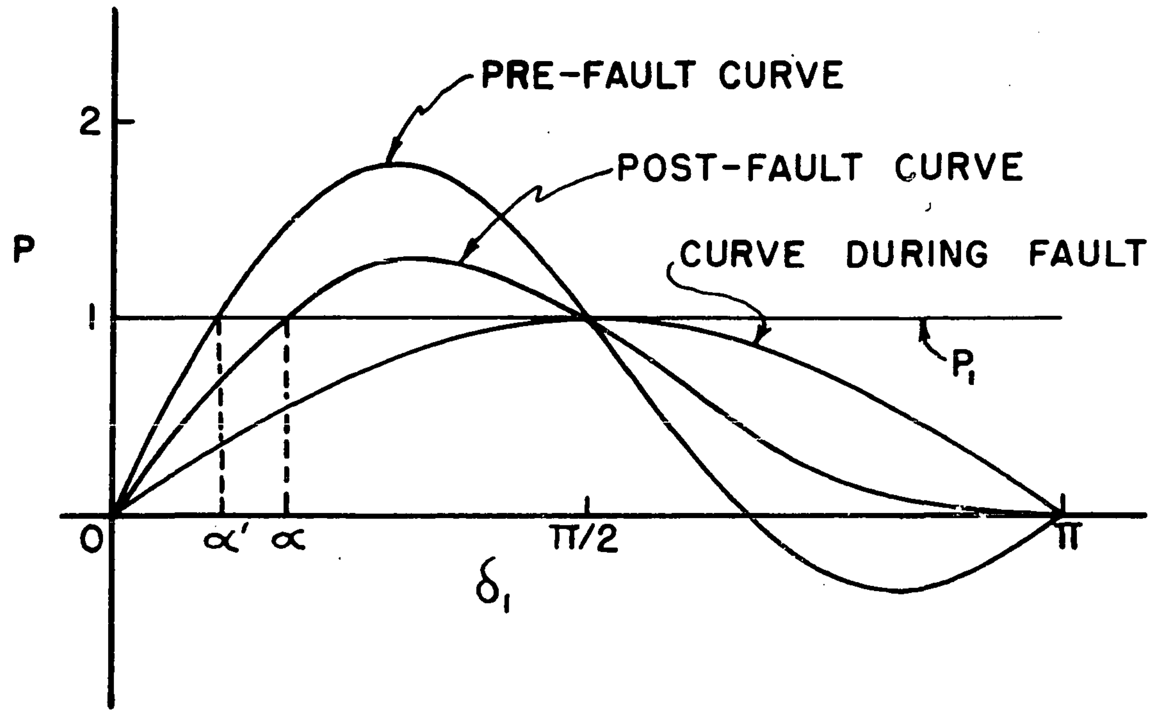

Fig. 3b. Power-angle curves for a symmetrical three-machine system

For the symmetrical system let

$$M_1 = M_2 = M_3 = 1$$

$$E_1 = E_2 = E_3 = 1$$

$$Y_1 = Y_2 = Y_3 = 1$$

$$P_1 = -P_3 = 1$$

$$P_2 = 0$$

If disturbances are introduced by changing only Y_3 in this system the symmetry will be maintained and $\delta_1 = -\delta_3$, $\delta_2 = 0$, $d\delta_2/dt = d^2\delta_2/dt^2 = 0$. With these conditions equations 24 become

$$\frac{d^2\delta_1}{dt^2} = 1 - \sin\delta_1 - \sin 2\delta_1$$

$$0 = 0 - \sin(-\delta_1) - \sin\delta_1 \qquad 25$$

$$-\frac{d^2\delta_1}{dt^2} = -1 - \sin(-2\delta_1) - \sin(-\delta_1)$$

Using the relation $\sin x = -\sin(-x)$ equations 25 become

$$\frac{d^2\delta_1}{dt^2} = 1 - \sin\delta_1 - \sin 2\delta_1$$

$$\qquad 26$$

$$\frac{d^2\delta_1}{dt^2} = 1 - \sin 2\delta_1 - \sin\delta_1$$

Making the substitutions

$$\frac{d\delta_1}{dt} = \frac{dx}{dt} = \omega \qquad 27$$

$$\delta_1 = x + \alpha \qquad 28$$

$$\frac{d^2\delta_1}{dt^2} = \frac{d\omega}{dt} \qquad 29$$

where α is the stable equilibrium value of δ_1, equations 26 may be written as

$$\frac{dx}{dt} = \omega \qquad 30$$

$$\frac{d\omega}{dt} = 1 - \sin(x + \alpha) - \sin(2x + 2\alpha) \qquad 31$$

The phase trajectory is obtained by dividing equation 31 by corresponding sides of 30 and integrating

$$\frac{d\omega/dt}{dx/dt} = \frac{1 - \sin(x + \alpha) - \sin(2x + 2\alpha)}{\omega} \qquad 32$$

$$\frac{\omega^2}{2} = x + \cos(x + \alpha) + \frac{1}{2}\cos(2x + 2\alpha) + C'' \qquad 33$$

Equation 33 may also be obtained by assuming a Liapunov function

$$V = \frac{\omega^2}{2} + \int_0^x \left[-1 + \sin(x' + \alpha) + \sin(2x' + 2\alpha)\right] dx' \qquad 34$$

and performing the integration. Equation 34 should be compared with 15 for the one-machine system. The integrand of equation 34 is the negative of the acceleration in equations 26. The presence of the double angle term is due to the nature of the system.

Now consider the line from 1 to 3 as open for the faulted condition and with $Y_3 = 1/2$ for the post-fault condition. The corresponding power-angle curves are given in

Fig. 3b. They are no longer simple sine curves because of the sin 2δ term. Steady state operation before the fault is at $\delta_1 = 20^{o}21'$ as obtained by setting $d^2\delta_1/dt^2 = 0$ in equation 26 or from the intersection of P_1 and the pre-fault curve.

Fig. 4a illustrates typical phase trajectories for the system in the post-fault condition. In Fig. 4b typical trajectories for stable and unstable clearing times are shown. It is evident that the trajectories for this special symmetrical case are much like those of the equivalent two-machine case even though the power-angle curves are somewhat different.

The next three-machine system to be studied is not symmetrical and the manipulation of the equations will be simplified somewhat if the notation for the time derivative of the form dx/dt is written $\dot{x}$. Other variables will be treated in a like manner and the second derivative becomes $\ddot{x}$. Using the new notation and making the following substitutions:

$$\delta_1 = x_1 + \alpha_1$$

$$\delta_2 = x_2 + \alpha_2$$

$$\delta_3 = x_3 + \alpha_3 \qquad 35$$

$$E_1E_2Y_1 = K_1$$

$$E_2E_3Y_2 = K_2$$

$$E_1E_3Y_3 = K_3$$

equations 24 become

$$\dot{x}_1 = \omega_1$$

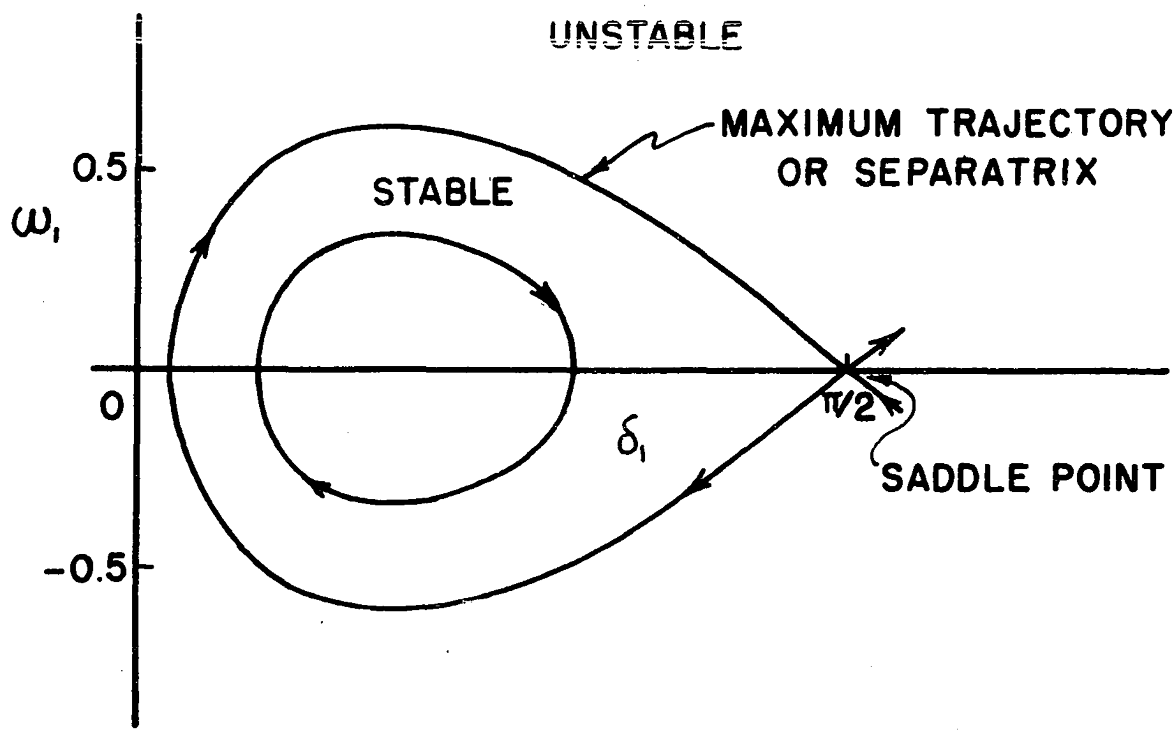

Fig. 4a. Phase trajectories of a symmetrical three-machine system, post-fault conditions

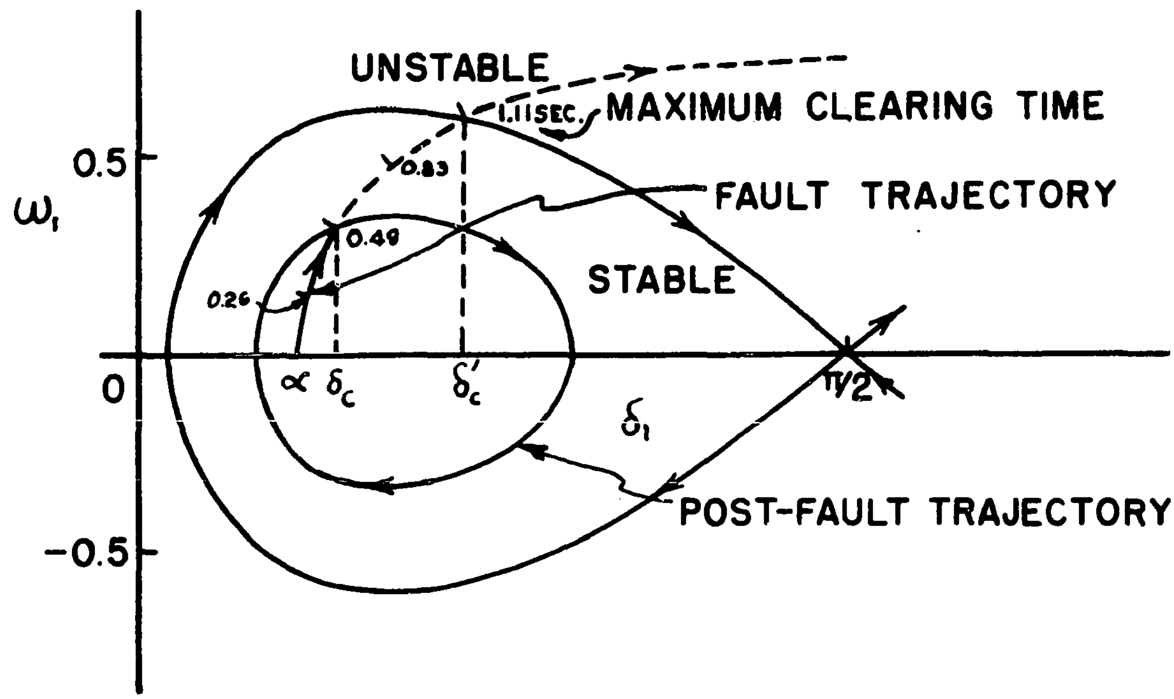

Fig. 4b. Phase trajectories of a symmetrical three-machine system during and after clearing a fault

$$M_1\dot{\omega}_1 = P_1 - K_1 \sin(x_1 - x_2 + \alpha_1 - \alpha_2)$$
$$- K_3 \sin(x_1 - x_3 + \alpha_1 - \alpha_3)$$
$$\dot{x}_2 = \omega_2$$
$$M_2\dot{\omega}_2 = P_2 - K_1 \sin(x_2 - x_1 + \alpha_2 - \alpha_1) \qquad 36$$
$$- K_2 \sin(x_2 - x_3 + \alpha_2 - \alpha_3)$$
$$\dot{x}_3 = \omega_3$$
$$M_3\dot{\omega}_3 = P_3 - K_3 \sin(x_3 - x_1 + \alpha_3 - \alpha_1)$$
$$- K_2 \sin(x_3 - x_2 + \alpha_3 - \alpha_2)$$

where the α_i are the stable equilibrium values of the δ_i.

The Liapunov function for the general three-machine case does not follow directly from the form for the single machine system. However, the negative integrals of the V-function given below are the areas under the power input curves for the three machines and the positive integrals are the areas under the power-angle curves for the interconnecting admittances.

After considerable manipulation of various choices, the Liapunov function chosen for the general three-machine system is

$$V = M_1 \frac{\omega_1^2}{2} + M_2 \frac{\omega_2^2}{2} + M_3 \frac{\omega_3^2}{2}$$
$$- \int_0^{x_1} P_1 dx_1' - \int_0^{x_2} P_2 dx_2' - \int_0^{x_3} P_3 dx_3'$$

$$+\int_0^{(x_1 - x_2)} K_1 \sin(x_1' - x_2' + \alpha_1 - \alpha_2)\, d(x_1' - x_2') \qquad 37$$

$$+\int_0^{(x_2 - x_3)} K_2 \sin(x_2' - x_3' + \alpha_2 - \alpha_3)\, d(x_2' - x_3')$$

$$+\int_0^{(x_3 - x_1)} K_3 \sin(x_3' - x_1' + \alpha_3 - \alpha_1)\, d(x_3' - x_1')$$

and has the value zero at the origin. It is always positive if the sum of the integral terms is greater than zero, i.e., the net restoring force for the system is positive. This condition is analogous to that for the one-machine case. The total time derivative of equation 37 is

$$\dot{V} = M_1\omega_1\dot{\omega}_1 + M_2\omega_2\dot{\omega}_2 + M_3\omega_3\dot{\omega}_3 - P_1\dot{x}_1 - P_2\dot{x}_2 - P_3\dot{x}_3$$

$$+ K_1(\dot{x}_1 - \dot{x}_2)\sin(x_1 - x_2 + \alpha_1 - \alpha_2)$$

$$+ K_2(\dot{x}_2 - \dot{x}_3)\sin(x_2 - x_3 + \alpha_2 - \alpha_3)$$

$$+ K_3(\dot{x}_3 - \dot{x}_1)\sin(x_3 - x_1 + \alpha_3 - \alpha_1) \qquad 38$$

After making substitutions for ω_i and $\dot{\omega}_i$ from equations 36 and rearranging, equation 38 becomes

$$\dot{V} = \dot{x}_1\big[P_1 - K_1\sin(x_1 - x_2 + \alpha_1 - \alpha_2)$$

$$- K_3\sin(x_1 - x_3 + \alpha_1 - \alpha_3)\big]$$

$$+ \dot{x}_2\big[P_2 - K_1\sin(x_2 - x_1 + \alpha_2 - \alpha_1)$$

$$- K_2\sin(x_2 - x_3 + \alpha_2 - \alpha_3)\big]$$

$$+ \dot{x}_3 \left[P_3 - K_3 \sin(x_3 - x_1 + \alpha_3 - \alpha_1) - K_2 \sin(x_3 - x_2 + \alpha_3 - \alpha_2)\right]$$

$$- \dot{x}_1 \left[P_1 - K_1 \sin(x_1 - x_2 + \alpha_1 - \alpha_2) + K_3 \sin(x_3 - x_1 + \alpha_3 - \alpha_1)\right]$$

$$- \dot{x}_2 \left[P_2 + K_1 \sin(x_1 - x_2 + \alpha_1 - \alpha_2) - K_2 \sin(x_2 - x_3 + \alpha_2 - \alpha_3)\right]$$

$$- \dot{x}_3 \left[P_3 - K_3 \sin(x_3 - x_1 + \alpha_3 - \alpha_1) + K_2 \sin(x_2 - x_3 + \alpha_2 - \alpha_3)\right] \qquad 39$$

If the relation sin(-x) = -sin x is applied to selected terms in equation 39 the result is

$$\dot{V} \equiv 0$$

Since the V function (equation 37) is positive definite and its time derivative is identically zero, the relation chosen is suitable to demonstrate the stability of the equilibrium. The next step is to examine the surface in the hyperspace to see if it is a boundary between the regions of stability and instability.

The system chosen for study is the three-machine network pictured in Fig. 3a. The constants for the differential equations 24 are, with all values in per unit,

$$M_1 = 0.02$$

$$M_2 = 0.002$$

$$M_3 = 0.03$$

$E_1 = E_2 = E_3 = 1$

$Y_1 = 2$

$Y_2 = 3$

$Y_3 = 1$

$P_1 = -P_3 = 1.5$

$P_2 = 0$

The power inputs to the various machines indicate that machine number one is delivering power, machine number three is absorbing the same amount and the second machine is acting as a synchronous condenser. The steady state values of the angles are $\delta_1 = 24.88^o$, $\delta_2 = 0$ and $\delta_3 = -16.25^o$.

Conditions following a fault are simulated by changing Y_2 from 3 to 1 and giving various velocities to the machines. The stable equilibrium point with the new value of Y_2 is at $\delta_1 = \alpha_1 = 18.78^o$, $\delta_2 = \alpha_2 = 0$ and $\delta_3 = \alpha_3 = -40.08^o$ where the α_i correspond to the notation used in equations 36. Using the same notation an unstable equilibrium is determined to be at $x_1 = 62.67^o$, $x_2 = 54.80^o$ and $x_3 = -21.38^o$. This equilibrium point corresponds to δ_m in Fig. 1c for the one-machine case. Substitution of the above values for x_i and α_i in equation 37 with the $\omega_i = 0$ gives the value of V as 0.428 for this system.

A computer program which utilized the Runge-Kutta-Gill method (50) was used to obtain the time solution of equations 24. The time solution in each case was carried out until stable or unstable behavior was established for the given set of initial conditions. Several rather typical sets of re-

Table 1. Effect of initial conditions on stability and comparison of actual and predicted borderline cases

Run no.	Predicted $\dot{\delta}_1$	Predicted $\dot{\delta}_3$	Actual $\dot{\delta}_1$	Actual $\dot{\delta}_3$	State
14			8.00	0.00	unstable
15			7.00	0.00	stable
17	5.59	0.00	7.75	0.00	borderline
25	0.00	-4.60	0.00	-7.50	borderline
26	3.95	-3.23	4.25	-3.45	borderline
35	4.74	-2.42	5.10	-2.60	borderline
37			2.80	-2.28	stable
38			3.42	-2.80	stable
41	2.12	-4.23	2.52	-5.05	borderline

sponses are given in Figs. 5 and 6. It is interesting to note the behavior of machine number two. Its frequency of oscillation is several times that of the other machines because of its small inertia constant. Close examination of the curves reveals that the frequency of the small machine is lower when the system is nearly or actually unstable. Machine two also tends to swing with machine one because of the closer coupling between them as compared to the coupling between two and three.

Initial values of several cases are given in Table 1. The initial values for the angles for all runs were taken as the steady state values $\delta_1 = 24.88^\circ$, $\delta_2 = 0$ and $\delta_3 = -16.25^\circ$. The predicted values of velocity necessary to produce a borderline case of stability were computed from equation 37 using the value of 0.428 for V.

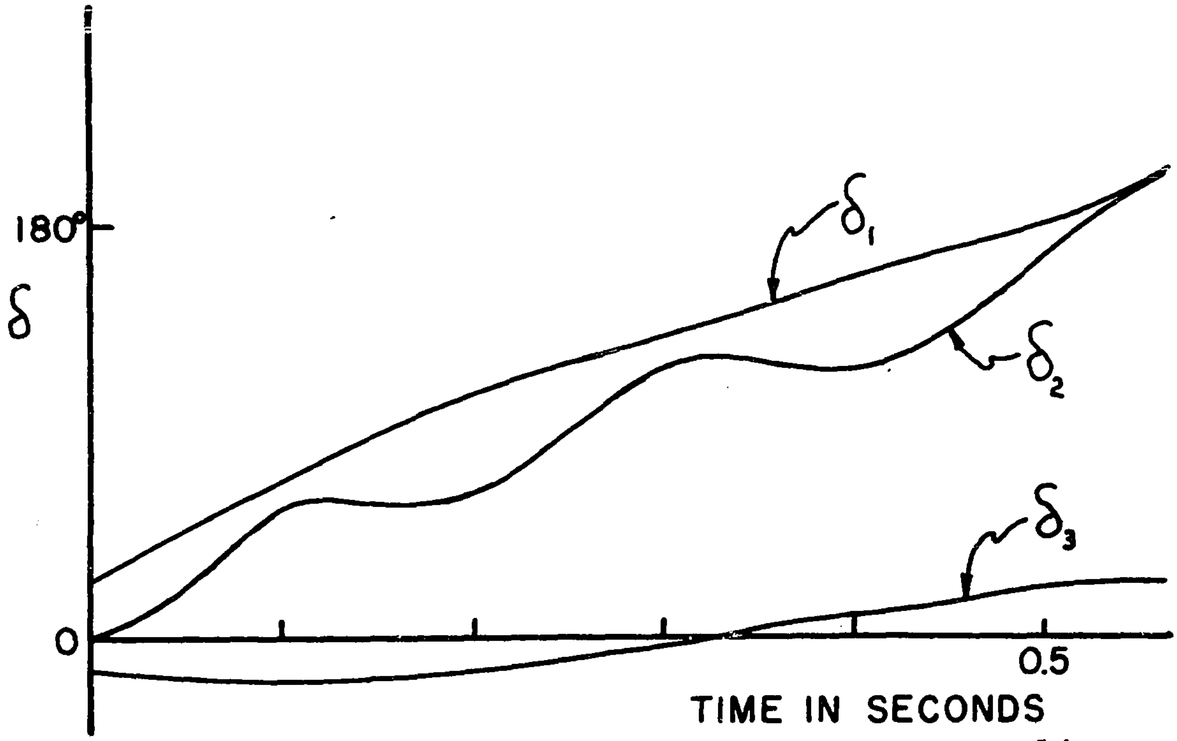

Fig. 5a. Swing curves for a three-machine system, run 14, unstable

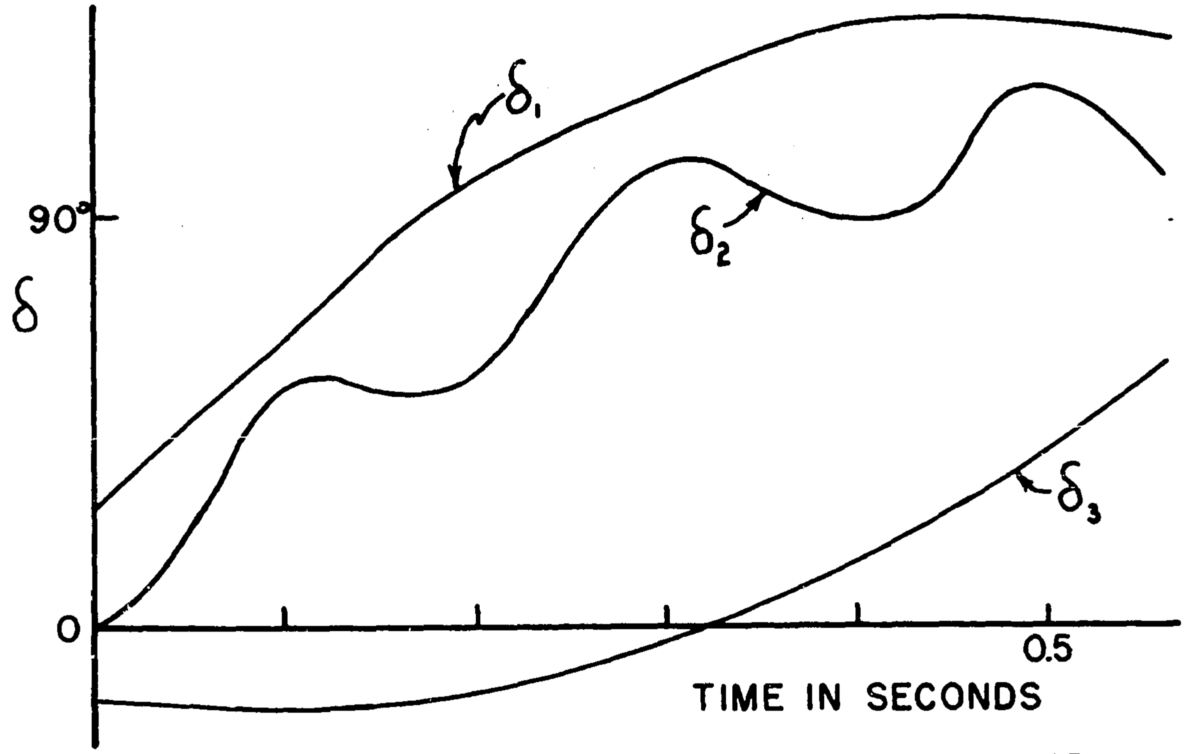

Fig. 5b. Swing curves for a three-machine system, run 15, stable

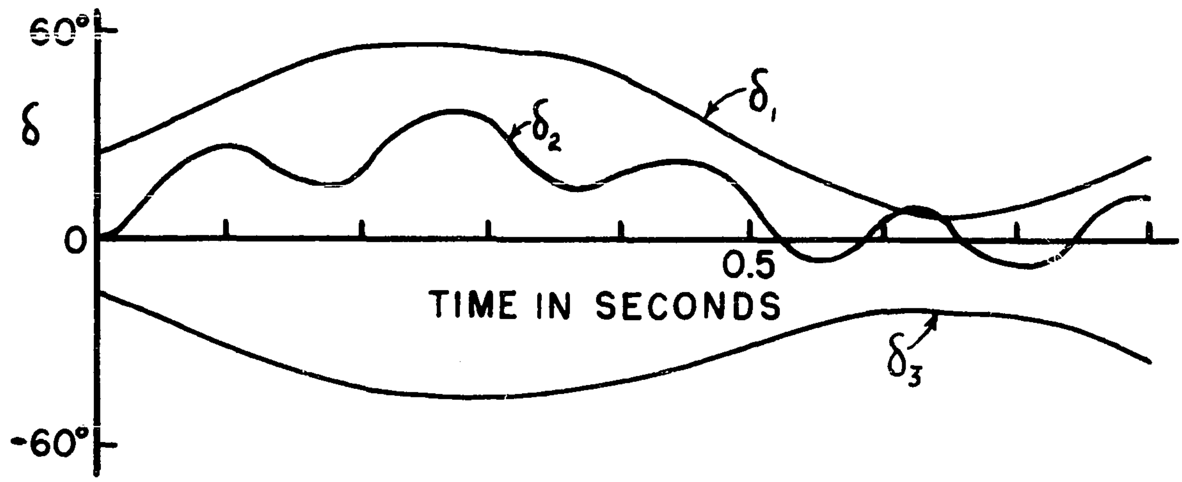

Fig. 6a. Swing curves for a three-machine system, run 37, V = 0.272, very stable

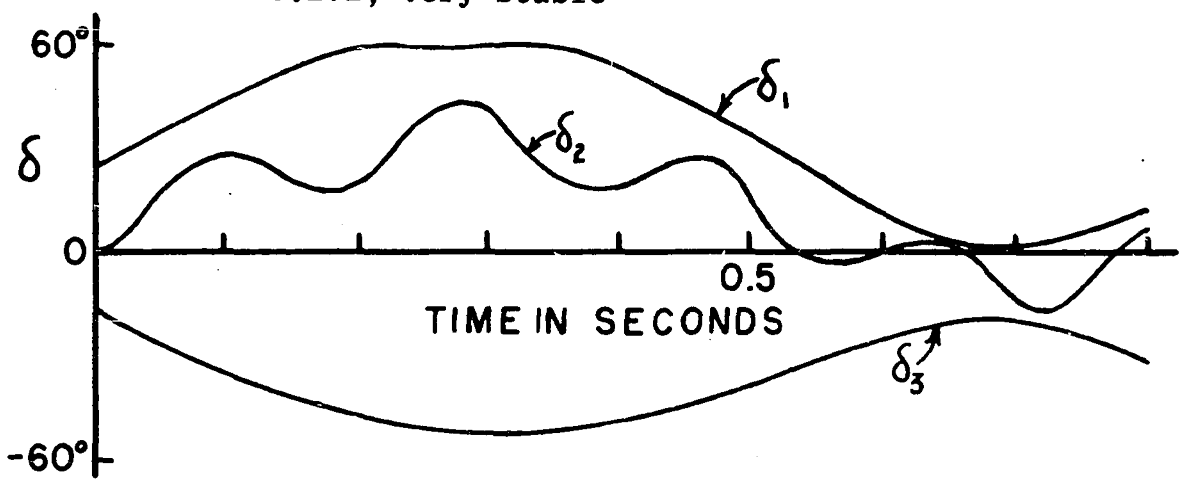

Fig. 6b. Swing curves for a three-machine system, run 38, V = 0.350, stable

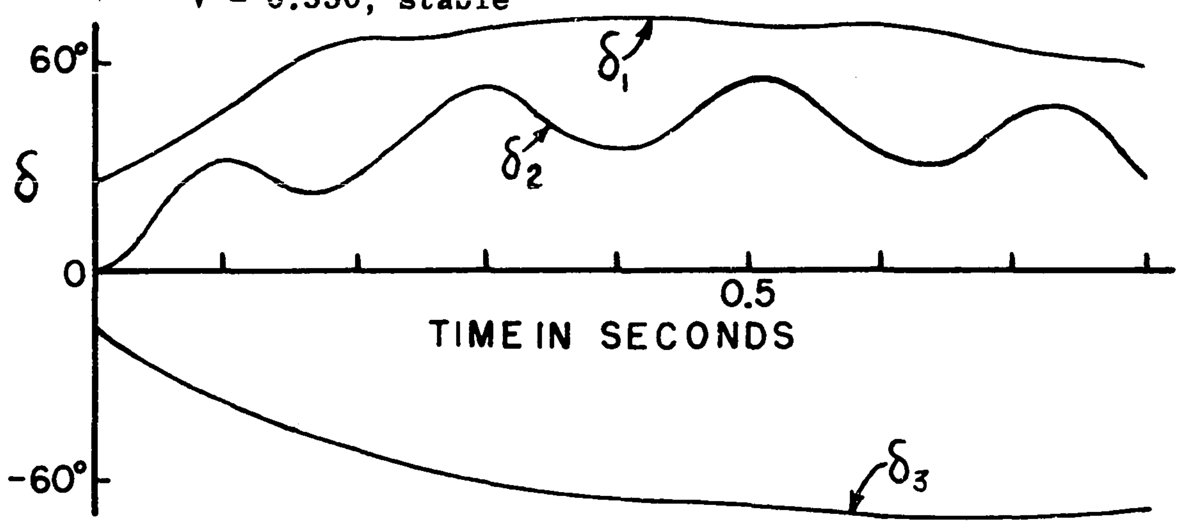

Fig. 6c. Swing curves for a three-machine system, run 26, V = 0.474, borderline

V. RESULTS AND CONCLUSIONS

In the one and two-machine problems the Liapunov function chosen gave the exact boundary between the stable and unstable regions. Examination of Table 1 indicates that the Liapunov method gives quite conservative results in runs 17 and 25 where all of the initial velocity is concentrated in one machine. If the initial velocities are distributed between the two large machines as in runs 26, 35 and 41 the actual and predicted values are much closer together with the predicted velocities still below the actual values for borderline stability. The direct method of Liapunov always assures a result which is conservative because the boundary of the stable region as defined by a proper Liapunov function is always inside or on the actual boundary of the stable region.

Even though the method in the case of three machines does not give an exact answer to whether the system is stable or unstable for the conditions assumed, this shortcoming is not as serious as it may first appear because the assumed conditions do not necessarily exist at the time of a fault or switching operation. It is evident then that no method can give an exact answer for stability under all conditions although some methods will certainly be better than others in a given situation.

The three sets of curves in Fig. 6 illustrate the comparison between the values of V and the relative stability of the system as shown by the time response. For V = 0.272, Fig. 6a, the system is quite stable. With V = 0.350 in the second set of curves the system is less stable as evidenced by the larger deviations of the machine angles. Fig. 6c illustrates the borderline case with V = 0.474. Thus a lower value of V does indicate a greater degree of stability.

Graham and McRuer (49, p. 368) point out that the Liapunov function chosen is often too restrictive. It appears here, however, that good prediction of stability may be expected if the kinetic energies are distributed among the machines. Such distribution is the usual condition in practical situations. As suggested by Graham and McRuer, a fruitful area for further research would be in the development of Liapunov functions which are less restrictive. Hahn (51, pp. 78-82) presents a method used by Zubov to determine exact stability boundaries for certain systems. The method failed to give results for the system under study here, but perhaps a different approach could be found.

The methods as developed in this dissertation apply only to lossless systems and have been checked for up to three machines. It would appear, however, that only a slight modification of the Liapunov function would be needed to account for the constant term added to the swing curve equation because of resistance losses. More practical applications would require the inclusion of losses and systems with more machines. A further limitation of the method is that the angular positions and velocities of all machines must be known at the instant of the final switching operation. If there is no fault and only one switching operation when the system is in the steady state, stability can be predicted as the angles and velocities are known. If a computer study is being made, computation may be stopped at the last instant of switching and the value of V computed to determine the relative stability rather than computing the remainder of the swing curves.

When the Liapunov method is applied to the stability problem, the time response is not needed beyond the point of application of the method. The engineer making the study thus loses some information about the behavior of the various machines in the time domain. He may know, however, that the system is stable under the conditions assumed and therefore the important question has been answered. In the case of the one-machine system, the trajectories in the phase plane present the response of the generator in a somewhat different manner than the swing curves. Perhaps a similar approach could be developed for the trajectories of the multi-machine system in a multidimensional phase space.

Since the Liapunov function is zero at the stable equilibrium and increases to a maximum at the boundary of the indicated stable region, the value of V at the last instant of switching indicates the relative stability of the system.

Thus the application of the Liapunov method to the problem of power system stability offers a new approach and gives further information on the behavior of such systems.

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VIII. APPENDIX

The basis of Liapunov's direct method as used in this dissertation is the Liapunov theorem as given by Hahn (51, p. 14) as theorem 4.1 (Given below with slightly different notation to correspond to that used in this study). The stability of the solutions of a set of differential equations

$$\frac{dx}{dt} = f(x,t) \qquad 40$$

and their dependence on the initial values is to be investigated. x and f are column vectors and f is continuous and of such a nature that the existence and uniqueness of the solutions, as well as their continuous dependence on the initial values is assured.

The Liapunov theorem as stated by Hahn is: The equilibrium is stable if there exists a positive definite function V(x,t) such that its total derivative dV/dt for the differential equation 40 is not positive.

The function V(x,t) is called a Liapunov function and has the following properties:

1. The function is positive definite.
2. It is a function of the time t and the n variables x_1, x_2, ·····, x_n of the system represented by equation 40.
3. dV/dt along the trajectories of the system is not positive.

The statement in Hahn's theorem that the total derivative of the Liapunov function, dV/dt, for the differential equation 40 is not positive has the following significance: When the total time derivative of the function V(x,t) is taken, terms involving dx_1/dt, dx_2/dt, $\cdots\cdots$, dx_n/dt arise and thus equation 40 is incorporated into dV/dt. The stability of the equilibrium is assured if dV/dt is non-positive.

A function V(x,t) is positive definite if V(0,t) = 0 and V(x,t) > 0 in a neighborhood of the origin. If the variable t does not appear explicitly in equation 40, then it does not appear in the Liapunov function chosen. There are no general rules for forming Liapunov functions although special methods have been developed for certain classes of functions. Hahn (51, p. 78ff) cites one such method by Zubov. The usual choice, because it is convenient mathematically, is to use a quadratic form. Widder (52) gives a quadratic form in three variables as

$$\begin{aligned} F(x_1,x_2,x_3) = {} & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 \\ & + a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 \\ & + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3^2 \end{aligned} \qquad 41$$

with $a_{ij} = a_{ji}$. In many special cases a quadratic form plus an appropriate integral will result in a useful Liapunov function. In view of the many choices available, a statement from Graham and McRuer (49, p. 369) is not inappropriate, "What passes for competence is a knowledge of previous fortunate experience".

In a simple system such as a spring and mass with the differential equation

$$\ddot{x} + D\dot{x} + x = 0 \tag{42}$$

the Liapunov function might be written as

$$V = \frac{\dot{x}^2}{2} + \int_0^x x' \; dx' \tag{43}$$

where V represents the total energy in this case. The derivative of V with respect to time is

$$\begin{aligned} \dot{V} &= \dot{x}\ddot{x} + x\dot{x} \\ &= \dot{x}(\ddot{x} + x) \end{aligned} \tag{44}$$

from equation 42

$$\ddot{x} + x = -D\dot{x} \tag{45}$$

substitution of 45 in 44 gives

$$\dot{V} = -D\dot{x}^2 \tag{46}$$

which is always negative or zero for D positive. Note that the substitution of equation 45 in 44 introduces the system equation into $\dot{V}$.

If the damping factor D is identically zero, $\dot{V}$ is zero and V is constant so that the system operates along a path of constant energy and is stable. If D is positive, $\dot{V}$ is negative and therefore V and the system energy decrease with time. Such a system is asymtotically stable as it will come to rest at a stable equilibrium. If D were to be negative, the energy would continually increase and the system would be unstable.